Comment on "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid"

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IN Ref. 1 the authors claim to have generated an exact solution of the Navier-Stokes equation, corresponding to the motion of a viscous incompressible fluid inside a hollow sphere set up by oscillations of the sphere about a diameter. Unfortunately, their result is not in fact an exact solution of the Navier-Stokes equation. For, if one follows the authors of Ref. 1 and, using their coordinate system, sets radial and meridional velocity components identically equal to zero, and takes the azimuthal velocity component $v_\phi = \psi(r, \theta, t)$, the complete Navier-Stokes equation,[2] which expresses conservation of momentum in all three coordinate directions, implies the three equations

$$\psi^2/r = (1/\rho)(\partial\hat{p}/\partial r) \tag{1}$$

$$\psi^2 \cot\theta/r = (1/\rho)(1/r)(\partial\hat{p}/\partial\theta) \tag{2}$$

$$\frac{\partial\psi}{\partial t} = \nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{\psi}{r^2\sin^2\theta}\right] \tag{3}$$

Equation (3) is seen to be Eq. (1) of Ref. 1, after the correction of what seems to be a printing error in the latter. The authors of Ref. 1 obtained a solution of this equation, but they neglected Eqs. (1) and (2) above. As it happens, there is no pressure function $\hat{p}(r, \theta, t)$ such that Eqs. (1) and (2) can be satisfied for the function ψ determined in Ref. 1. For, eliminating $\hat{p}$ between (1) and (2), one finds that these equations are inconsistent unless

$$(\partial/\partial\theta)(\psi^2/r) = (\partial/\partial r)(\psi^2 \cot\theta)$$

from which it follows that

$$(\partial\psi/\partial\theta) = r\cot\theta(\partial\psi/\partial r) \tag{4}$$

One can verify by substitution that the result of Ref. 1 does not satisfy Eq. (4). More simply but less directly, one can observe that the solution ψ found in Ref. 1 has the form

$$\psi = \sin\theta f_1(r, t)$$

and that a function of this form can satisfy Eq. (4) only if $f_1(r, t) = C(t)r$. This is not the $f_1(r, t)$ determined in Ref. 1. Thus, the result of Ref. 1 is not an exact solution of the Navier-Stokes equation; indeed, the velocity field determined there does not conserve momentum in the radial and meridional directions.

Nevertheless, the work of Ref. 1 can be useful, if it is properly interpreted. This is indicated by the quite acceptable agreement between experiment and theory shown in Fig. 2 of Ref. 1. In fact, Ref. 1 can be interpreted as a treatment of the first-order terms in a formal expansion of the velocity field in powers of Φ, the amplitude of the sphere's angular displacement. In Ref. 1, the motion of the sphere was characterized by stating that its angular velocity Ω was sinusoidal in time, i.e.,

$$\Omega = \Omega_0 \cos pt$$

This is of course equivalent to specifying the sphere's angular displacement $\hat{\phi}$ to be sinusoidal,

$$\hat{\phi} = \Phi \sin pt$$

In terms of the sphere's angular displacement, the boundary condition at the surface of the sphere is

$$v_\phi(a, \theta, t) = a\Omega\sin\theta = ap\Phi\sin\theta\cos pt$$

with all other velocity components zero. If the problem is nondimensionalized, using pa as reference velocity and a as a reference length, this condition becomes

$$v_\phi'(1, \theta, t') = \Phi\sin\theta\cos t' \tag{5}$$

where here and in the following a prime denotes a nondimensional variable. If now one enters the nondimensionalized Navier-Stokes equation, the conservation of mass condition of a solenoidal velocity field, and boundary condition (5) with the formal expansion

$$v_r' = \Phi^2 u_2 + \Phi^3 u_3 + \ldots \tag{6a}$$

$$v_\theta' = \Phi^2 v_2 + \Phi^3 v_3 + \ldots \tag{6b}$$

$$v_\phi' = \Phi w_1 + \Phi^2 w_2 + \ldots \tag{6c}$$

$$(p/\rho) = \Phi^2\pi_2 + \Phi^3\pi_3 + \ldots \tag{6d}$$

where each dependent variable is taken to be a function of r', θ, and t', one finds that this expansion is not obviously inconsistent, and that the lowest order terms, i.e., those of order Φ, imply

$$\frac{\partial w_1}{\partial t'} = \frac{1}{\gamma}\left[\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial w_1}{\partial r'}\right) + \frac{1}{r'^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w_1}{\partial\theta}\right) - \frac{w_1}{r'^2\sin^2\theta}\right],\quad w_1(1, \theta, t') = \sin\theta\cos t'$$

where, as in Ref. 1, $\gamma = pa^2/\nu$. In essence, this is the problem solved in Ref. 1.

Thus, while of course the convergence of expansion (6) requires investigation, the work of Ref. 1 may well be accurate for the limiting case in which the amplitude Φ of the sphere's oscillation tends to zero while γ is fixed. This seems to have been the application its authors actually had in mind. But, since the results of Ref. 1 are not really based on an exact solution of the Navier-Stokes equation, it may be dangerous to apply them in other situations.

References

[1] Aprahamian, R., Johnson, R. L., and Koval, L. R., "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid," *AIAA Journal*, Vol. 7, No. 9, Sept. 1969, pp. 1793–1796.

[2] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, Addison Wesley, Reading, Mass., 1959, p. 52.

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Reply by Authors to D. A. Lee

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